Chat session with ChatGPT 4o   Date December 8, 2024

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Header: "Measure-preserving probability spaces for dynamical systems, or equivalently (X, μ, T)"

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Section 1 (Definitions and Notation): 1. $(X, \mathcal{B}, \mu)$ = a measure space 2. $T : X \to X$ is measurable 3. T preserves μ: $\mu(T^{-1}A) = \mu(A) \quad \forall A \in \mathcal{B}$

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Section 2 (Examples): - $(X, \mu) = ([0,1], \lambda)$, $T(x) = 2x \mod 1$ - Circle rotation $(X, \mu) = (S^1, \lambda)$, $T(x) = x + \alpha \mod 1$

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Section 3 (Ergodicity and Mixing): Ergodicity: 1. T is **ergodic** if: - $\forall A \in \mathcal{B}$, $T^{-1}A = A \implies \mu(A) \in \{0, 1\}$

Mixing: 2. T is **mixing** if: - $\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B)$

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Section 4 (Applications of Mixing): - **Pointwise Ergodic Theorem**: - $f \in L^1(X, \mu)$, $\frac{1}{N}\sum_{n=1}^{N} f(T^n x) \to \int_X f \, d\mu \quad \mu$-almost everywhere.

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Section 5 (Further Notes and Symbols): 1. $B : L^2(X, \mu) \to L^2(X, \mu)$, where $B(f) = \int_X f d\mu$. 2. T generates a group $(T^n)_{n \in Z}$. 3. Ergodic systems: No nontrivial invariant subsets. 4. Mixing implies ergodicity, but not vice versa.

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